SIAM J. DISC. MATH.
Vol. 3, No. 4, pp. 502–523, November 1990

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INTEGER POLYHEDRA ARISING FROM CERTAIN NETWORK DESIGN PROBLEMS WITH CONNECTIVITY CONSTRAINTS*

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Abstract. In this paper a general integer linear programming model is presented for the important practical problem of designing minimum-cost survivable networks, and this model is related to concepts in graph theory and polyhedral combinatorics. In particular, several interesting special cases of this general model are considered, including the minimum spanning tree problem, the Steiner tree problem, and the minimum cost k-edge connected and k-node connected network design problems. The integer polyhedra associated with these problems are studied, those inequalities from natural ILP-formulations that define facets are identified, the separation problem for these facets is addressed, and how good lower bounds can be obtained from the models studied here is indicated.

Key words. network design, network survivability, connectivity, integer programming, polyhedral combinatorics

AMS(MOS) subject classifications. 05C40, 90C27, 90C50

1. Introduction. For over thirty years, mathematical models arising from the design and analysis of communication networks have been a major focal point for research efforts in the fields of operations research, graph theory, and discrete mathematics. This fertile area has been stimulated by the great practical importance of the associated real-world problems and the wide range applicability of these models on the one hand, and the interesting structural and algorithmic questions and the elegant theoretical results on the other hand. The introduction of new types of networks and new technologies has resulted in a rich variety of models which have been studied over the years.

A recent trend in communication networks is the emergence of fiber optic technology as one of the major components in the "network of the future." This transmission medium is cost effective, reliable, and provides very high transmission capacity. This combination promises to usher in new telecommunication services requiring large amounts of bandwidth. At the same time, the unique characteristics of this technology imply the need for new network design approaches.

Survivability is an important factor in the design of communication networks. Network survivability is used here to mean the ability to restore service in the event of a catastrophic failure of a network component, such as the complete loss of a transmission link or the failure of a switching node. Service could be restored by routing traffic through other existing network links and nodes, assuming that the design of the network has provided for this additional connectivity. Clearly, a higher level of redundant connectivity results in a greater network survivability and a greater overall network cost. This leads to the problem of designing a minimum-cost network which meets certain required connectivity constraints.

Survivability is a particularly important issue for fiber networks. The high capacity of fiber facilities results in much more sparse network designs with larger amounts of traffic carried by each link than is the case with traditional bandwidth-limited technologies. This increases the potential damage to network services due to link or node failures. It is necessary to trade off the potential for lost revenues and customer goodwill against the

* Received by the editors December 21, 1988; accepted for publication October 10, 1989.

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extra costs required to increase the network survivability. Recent work on methods for designing survivable fiber communication networks by Monma and Shallcross (1989) concludes that (1) survivability is an important issue for fiber networks, (2) "two-connected" topologies provide a high level of survivability in a cost effective manner, and (3) good heuristic methods exist for quickly generating "near-optimal" networks.

In §§ 2 and 3 of this paper, we present a general integer linear programming model for the network design problem with connectivity constraints and relate this model to concepts in graph theory and polyhedral combinatorics. In the remaining sections we consider important special cases of this general model and study the associated integer polyhedra, identify which natural inequalities define facets, address the separation problem for these facets, and indicate how good lower bounds can be obtained from these models. Section 4 is concerned with the minimum spanning tree problem where a complete linear description of the associated integer polyhedron is given. This result follows easily from matroid theory (see Edmonds (1971)) and is used in later sections. Related work for the Steiner tree problem is described in § 5. Sections 6 and 7 examine the minimum-cost network design problems with edge connectivity constraints and node connectivity constraints, respectively, from a polyhedral point of view. A model which combines both edge and node connectivity constraints is considered in § 8.

2. A general model. In this section, we formalize the network design problems which are being considered in this paper. A set V of *nodes* is given which represents the locations of the switches (offices) which must be interconnected into a network in order to provide the desired services. A collection E of *edges* is also specified which represents the possible pairs of nodes between which a direct transmission link can be placed. We let $G = (V, E)$ be the (undirected) graph of possible direct link connections. Each edge $e \in E$ has a nonnegative *fixed cost* c_e of establishing the direct link connection. The graph G may have parallel edges but contains no loops. (Thus we assume throughout this paper that all graphs considered are loopless but may have parallel edges. Graphs without parallel edges are called *simple*.) The cost of establishing a network consisting of a subset $F \subseteq E$ of edges is the sum of the costs of the individual links contained in F. The goal is to build a minimum-cost network so that the required survivability conditions, which we describe below, are satisfied. We note that the cost here represents setting up the topology for the communication network and includes placing conduits in which to lay the fiber cables, placing the cables into service, and other related costs. We do not consider costs which depend on how the network is implemented such as routing or multiplexing, nor do we consider repeater costs. Although these costs are also important, it is usually the case that a topology is first designed and then these other costs are considered in a second stage of optimization.

For any pair of distinct nodes $s, t \in V$, an $[s, t]$*-path* P is a sequence of nodes and edges $(v_0, e_1, v_1, e_2, \cdots, v_{l-1}, e_l, v_l)$, where each edge e_i is incident with the nodes v_{i-1} and v_i $(i = 1, \cdots, l)$, where $v_0 = s$ and $v_l = t$, and where no node or edge appears more than once in P. A collection $P_1, P_2, \cdots, P_k$ of $[s, t]$-paths is called *edge-disjoint* if no edge appears in more than one path, and is called *node-disjoint* if no node (except for s and t) appears in more than one path. (Remark: In order to be consistent with standard graph theory we do not consider two parallel edges as two node disjoint paths.)

The survivability conditions require that the network satisfy certain edge and node connectivity requirements. In particular, for each pair of distinct nodes $s, t \in V$, three nonnegative integers r_{st}, k_{st}, and d_{st} are given. The numbers r_{st} represent the *edge survivability requirements* and the numbers k_{st} and d_{st} the *node survivability requirements*, meaning that the network $N = (V, F)$ to be designed has to have the property that, for each pair $s, t \in V$ of distinct nodes, N must contain at least r_{st} edge disjoint $[s, t]$-paths and the removal of at most k_{st} nodes (different from s and t) from N must leave at least d_{st} edge disjoint $[s, t]$-paths. (Clearly, we may assume that $k_{st} \leqq |V| - 2$ for all $s, t \in V$, and we will do this throughout this paper.) These conditions ensure that some communication path between s and t will survive a prespecified level of combined failures of both nodes and links. The levels of survivability specified depend on the relative importance placed on maintaining connectivity between different pairs of offices.

Let us now introduce, for each edge $e \in E$, a variable x_e and consider the vector space $\mathbf{R}^E$. Every subset $F \subseteq E$ induces an *incidence vector* $\chi^F = (\chi_e^F)_{e \in E} \in \mathbf{R}^E$ by setting $\chi_e^F \equiv 1$ if $e \in F$, $\chi_e^F \equiv 0$ otherwise, and vice versa, each 0/1-vector $x \in \mathbf{R}^E$ induces a subset $F^x \equiv \{e \in E \mid x_e = 1\}$ of the edge set E of G. If we speak of the incidence vector of a path in the sequel we mean the incidence vector of the edges of the path. We can now formulate the network design problem introduced above as the following integer linear program.

$$\text{(2.1)} \qquad \min \sum_{ij \in E} c_{ij} x_{ij}$$

subject to

$$\begin{array}{lll}
\text{(i)} & \displaystyle\sum_{i \in W} \sum_{j \in V \setminus W} x_{ij} \geqq r_{st} & \text{for all pairs } s, t \in V,\ s \neq t \text{ and} \\
& & \text{for all } W \subseteq V \text{ with } s \in W,\ t \notin W; \\
\text{(ii)} & \displaystyle\sum_{i \in W} \sum_{j \in V \setminus (Z \cup W)} x_{ij} \geqq d_{st} & \text{for all pairs } s, t \in V,\ s \neq t \text{ and} \\
& & \text{for all } Z \subseteq V \setminus \{s, t\} \text{ with } |Z| = k_{st} \text{ and} \\
& & \text{for all } W \subseteq V \setminus Z \text{ with } s \in W,\ t \notin W; \\
\text{(iii)} & 0 \leqq x_{ij} \leqq 1 & \text{for all } ij \in E; \\
\text{(iv)} & x_{ij} \text{ integral} & \text{for all } ij \in E.
\end{array}$$

Note that if, for each pair s, t of distinct nodes in V and for each set $Z \subseteq V \setminus \{s, t\}$ with $|Z| = k_{st}$, $N - Z$ contains at least d_{st} edge disjoint $[s, t]$-paths, then all node survivability requirements are satisfied, i.e., inequalities of type (ii) need not be considered for node sets $Z \subseteq V \setminus \{s, t\}$ with $|Z| < k_{st}$. It follows from Menger's theorem that, for every feasible solution x of (2.1), the subgraph $N = (V, F^x)$ of G defines a network that satisfies the given edge and node survivability requirements.

This formulation is quite general and, as far as we know, this mixture of node and edge survivability requirements has not been considered in the published literature. (The survey paper by Christofides and Whitlock (1981) discusses some related models.) Problem (2.1) is NP-hard as it contains various NP-hard special cases. Some of them will be mentioned in the sequel.

The classical *network synthesis problem* for multiterminal flows is obtained from (2.1) by dropping constraints (ii) and (iv). In the standard formulation of the network synthesis problem the upper bounds $x_e \leqq 1$ are not present. But our model allows parallel edges in the underlying direct-link graph, or equivalently, allows the upper bound in constraints (iii) to take on any nonnegative integer values for each edge. This linear programming problem has a number of constraints that is exponential in the number of nodes of G. For the case $c_{ij} = c$ for all $ij \in E$, where c is a constant, Gomory and Hu (1961) found a simple algorithm for its solution. Bland, Goldfarb, and Todd (1981) pointed out that the separation problem for the class of inequalities (i) can be solved in polynomial time by computing a minimum capacity cut; it thus follows by the ellipsoid method that the classical network synthesis problem can be solved in polynomial time.

The *minimum spanning tree problem* can be phrased as the task to find a minimum cost connected subset $F \subseteq E$ of edges that span V. (See Graham and Hell (1985) for a survey of work on this problem.) This problem can be viewed as a special case of (2.1) as follows. We drop constraints (ii) and set, for all distinct $s, t \in V$, $r_{st} \equiv 1$ in constraints (i). In § 4 we present a complete linear description of the integer polytope for the minimum spanning tree problem which follows easily from matroid theory (see Edmonds (1971)).

Similarly, the closely-related *Steiner tree problem* is to find a minimum cost connected subset $F \subseteq E$ of edges which span a specified subset $S \subseteq V$ of nodes. This problem is a special case of (2.1) where we drop constraints (ii) and set in constraints (i), $r_{st} \equiv 1$ for all $s, t \in S$, and $r_{st} \equiv 0$ otherwise. This problem is NP-hard and will be discussed in § 5.

A graph with at least two nodes is *k-edge* (respectively, *k-node*) *connected* if there are k edge-disjoint (respectively, node-disjoint) paths between every pair of distinct nodes. For our purposes, the graph K_1 consisting of just one node is k-edge and k-node connected for every natural number k. For a graph $G \neq K_1$, the largest integer k such that G is k-edge connected (respectively, k-node connected) is denoted by $\lambda(G)$ (respectively, $\kappa(G)$) and is called the *edge connectivity* (respectively, *node connectivity*) of G. Our definition, for instance, implies that, for a graph G with $n \geqq 2$ nodes such that every two nodes are linked by p edges, $\kappa(G) = n - 1$ and $\lambda(G) = p(n-1)$ holds.

There have been many papers in the graph theory literature that study properties of k-edge or k-node connected graphs. The problem of finding an optimal k-edge connected network is a special case of (2.1) where all inequalities (ii) are dropped and where, for all distinct $s, t \in V$, $r_{st} = k$. The problem of finding an optimal k-node connected network, $k \leqq |V| - 1$, is a special case of (2.1) as follows. We drop the constraints (i) and set, for all distinct $s, t \in V$, $k_{st} \equiv k - 1$ and $d_{st} \equiv 1$.

Monma, Munson, and Pulleyblank (1985) consider the *minimum cost* 2-*connected network design problem* where the underlying graph $G = (V, E)$ is a complete graph and the costs satisfy the triangle inequality, i.e., $c_{ik} \leqq c_{ij} + c_{jk}$ for all $i, j, k \in V$. They show that, for $k = 2$, there exists an optimal k-edge (respectively, k-node) connected solution whose nodes all have degrees k or $k + 1$, and such that the removal of any $1, 2, \cdots$, or k edges does not result in all connected components still being k-edge (respectively, k-node) connected. (This is extended to arbitrary $k \geqq 2$ by Bienstock, Brickell, and Monma (1987).) They also show that these conditions "characterize" the optimal 2-connected networks in an appropriate sense, but that this is not the case for $k \geqq 3$. We return to the k-connected network design problems in §§ 6 and 7.

3. The polyhedral approach. The main objective of this paper is to study the network design problem from a polyhedral point of view to see which of the inequalities (i), (ii), (iii) of (2.1) (and which of further classes of inequalities) are suitable choices for a cutting plane approach, i.e., we want to find a tighter LP-relaxation of the IP (2.1) than the one following from (2.1) by dropping the integrality constraints (iv). To do this, we define the following polytope. Let $G = (V, E)$ be a graph, let $E_V \equiv \{st \mid s, t \in V, s \neq t\}$, and let $r, k, d \in \mathbf{Z}_+^{E_V}$ be given. Then

$$(3.1) \qquad \mathrm{CON}\,(G; r, k, d) \equiv \mathrm{conv}\,\{x \in \mathbf{R}^E \mid x \text{ satisfies (i)}, \cdots, \text{(iv) of (2.1)}\}$$

is the polytope associated with the network design problem given by G and the edge and node survivability requirements r, k, and d. (Above "conv" denotes the convex hull operator.) In the sequel we will study CON $(G; r, k, d)$ for various special choices of r, k, and d. Let us mention here a few general properties of CON $(G; r, k, d)$ that are easy to derive.

Let $G=(V, E)$, r, k, $d \in \mathbf{Z}_+^{E_V}$ be given as above. Extending the terminology of Bollobas (1978), we say that $e \in E$ is *essential with respect to* $(G; r, k, d)$ (short: $(G; r, k, d)$-*essential*) if CON $(G-e; r, k, d)=\emptyset$. In other words, e is essential with respect to $(G; r, k, d)$ if its deletion from G results in a graph such that at least one of the survivability requirements cannot be satisfied. We denote the set of edges in E that are essential with respect to $(G; r, k, d)$ by ES $(G; r, k, d)$. Clearly, for all subsets $F \subseteq E \backslash$ES $(G; r, k, d)$, ES $(G; r, k, d) \subseteq$ ES $(G-F; r, k, d)$ holds. Let dim (S) denote the *dimension* of a set $S \subseteq \mathbf{R}^n$, i.e., the maximum number of affinely independent elements in S minus 1.

THEOREM 3.2. *Let* $G=(V, E)$ *be a graph and* r, k, $d \in \mathbf{Z}_+^{E_V}$ *such that* CON $(G; r, k, d) \neq \emptyset$. *Then*

$$\mathrm{CON}\,(G; r, k, d) \subseteq \{x \in \mathbf{R}^E \mid x_e=1 \quad \textit{for all } e \in \mathrm{ES}\,(G; r, k, d)\},$$

$$\dim\,(\mathrm{CON}\,(G; r, k, d))=|E|-|\mathrm{ES}\,(G; r, k, d)|.$$

Proof. If $e \in E$ is $(G; r, k, d)$-essential then every vector $x \in$ CON $(G; r, k, d)$ clearly satisfies $x_e=1$. So CON $(G; r, k, d)$ is contained in the affine space of dimension $|E|-|$ES $(G; r, k, d)|$ defined by the equations $x_e=1$, $e \in$ ES $(G; r, k, d)$. Let $a^T x=\alpha$ be an equation satisfied by all points in CON $(G; r, k, d)$. We may assume that $a_e=0$ for all $e \in$ ES $(G; r, k, d)$. If $e \in E \backslash$ES $(G; r, k, d)$ then the incidence vectors of E and $E \backslash \{e\}$ are contained in CON $(G; r, k, d)$, and thus $a^T \chi^E=a^T \chi^{E \backslash \{e\}}=\alpha$ holds. This implies $a_e=0$ for all $e \in E \backslash$ES $(G; r, k, d)$ and hence $a=0$. Therefore, dim CON $(G; r, k, d)=|E|-|$ES $(G; r, k, d)|$. $\square$

An inequality $a^T x \leqq \alpha$ is *valid* with respect to a polyhedron P if $P \subseteq \{x \mid a^T x \leqq \alpha\}$; the set $F_a \equiv \{x \in P \mid a^T x=\alpha\}$ is called the *face* of P defined by $a^T x \leqq \alpha$. If dim $(F_a)=$ dim $(P)-1$ and $F_a \neq 0$ then F_a is a *facet* of P and $a^T x \leqq \alpha$ is called *facet-defining* or *facet-inducing*.

THEOREM 3.3. *Let* $G=(V, E)$ *be a graph and* r, k, $d \in \mathbf{Z}_+^{E_V}$ *such that* CON $(G; r, k, d) \neq \emptyset$. *Then*

(a) $x_e \leqq 1$ *defines a facet of* CON $(G; r, k, d)$ *if and only if* $e \in E \backslash$ES $(G; r, k, d)$;

(b) $x_e \geqq 0$ *defines a facet of* CON $(G; r, k, d)$ *if and only if* $e \in E \backslash$ES $(G; r, k, d)$ *and* ES $(G; r, k, d)=$ ES $(G-e; r, k, d)$.

Proof. By Theorem (3.2), none of the inequalities $0 \leqq x_e \leqq 1$, $e \in$ ES $(G; r, k, d)$, defines a facet of CON $(G; r, k, d)$.

(a) If $e \in E \backslash$ES $(G; r, k, d)$ then the incidence vectors of E and $E \backslash \{f\}$, for each $f \in E \backslash($ES $(G; r, k, d) \cup \{e\})$, satisfy $x_e \leqq 1$ with equality and are linearly independent. Thus $x_e \leqq 1$ defines a facet of CON $(G; r, k, d)$.

(b) Suppose $e \in E \backslash$ES $(G; r, k, d)$. If there is an edge $f \in$ ES $(G-e; r, k, d) \backslash$ ES $(G; r, k, d)$ then, for all $x \in$ CON $(G; r, k, d)$, $x_e=0$ implies $x_f=1$; thus $x_e \geqq 0$ does not define a facet of CON $(G; r, k, d)$. If ES $(G; r, k, d)=$ ES $(G-e; r, k, d)$ then let $a^T x \geqq \alpha$ be a facet-defining inequality satisfied by all

$$x \in F_e \equiv \{x \in \mathrm{CON}\,(G; r, k, d) \mid x_e=0\}.$$

By Theorem 3.2 we may assume that $a_g=0$ for all $g \in$ ES $(G; r, k, d)$. Let $f \in E \backslash($ES $(G; r, k, d) \cup \{e\})$, then the incidence vectors of $E \backslash \{e\}$ and $E \backslash \{e, f\}$ are by assumption in CON $(G; r, k, d)$ and satisfy $a^T x=\alpha$. This implies $a_f=0$. It follows that $a^T x \geqq \alpha$ is a positive multiple of $x_e \geqq 0$, and thus, $x_e \geqq 0$ defines a facet of CON $(G; r, k, d)$. $\square$

Theorems 3.2 and 3.3 solve the dimension problem and characterize the trivial facets. However, these characterizations are (in a certain sense) algorithmically intractible as the next observation shows.

Remark 3.4. The following three problems are NP-hard.

Instance: A graph $G = (V, E)$ and vectors $r, k, d \in \mathbf{Z}_+^{E_V}$.

Question 1: Is CON $(G; r, k, d)$ empty?

Question 2: Is $e \in E$ an element of ES $(G; r, k, d)$?

Question 3: What is the dimension of CON $(G; r, k, d)$?

Proof. Clearly, if we have shown that the first problem is NP-hard, the definition of "essential" and Theorem 3.2 immediately imply that the other two problems are difficult as well.

The NP-hardness of Question 1 follows directly from a recent result of Ling and Kameda (1987). They proved that the following problem is NP-complete.

Instance: A simple graph $G = (V, E)$; two nodes $u, v \in V$, $u \neq v$; two nonnegative integers a and b.

Question: Does there exist a subset $Z \subseteq V$ with $|Z| = a$ and $u, v \notin Z$ such that $G - Z$ contains at most b edge disjoint $[u, v]$-paths?

Suppose we could determine in polynomial time whether CON $(G; r, k, d)$ is empty for the following (very special) choice of $r, k, d \in \mathbf{Z}_+^{E_V}$; $r \equiv 0$; $k_{uv} \equiv a$ and $k_{st} = 0$ otherwise; $d_{uv} \equiv b + 1$ and $d_{st} = 0$ otherwise. Then we could obviously answer the above question in polynomial time. $\square$

However, for most cases of practical interest in the design of optical fiber networks, the sets ES $(G; r, k, d)$ of essential edges can be determined easily, and thus the trivial LP-relaxation of (2.1) following from (3.2) and (3.3) can be set up without difficulties. (We will comment on this in the sequel.)

In fact, if we can determine ES $(G; r, k, d)$, we can decompose min $c^T x, x \in$ CON $(G; r, k, d)$ into several subproblems as follows. If $G^1, \cdots, G^p$ are the components of $G -$ ES $(G; r, k, d)$ then it is possible to compute vectors r^i, k^i, d^i $(i = 1, \cdots, p)$ such that G^i contains no $(G^i; r^i, k^i, d^i)$-essential elements and such that the incidence vector of $F \equiv F^{x^1} \cup F^{x^2} \cup \cdots \cup F^{x^p} \cup$ ES $(G; r, k, d)$ is an optimum solution of min $c^T x, x \in$ CON $(G; r, k, d)$, where x^i is an optimum solution of min $(c^i)^T x$, $x \in$ CON $(G^i; r^i, k^i, d^i)$ and c^i is a projection of c into an appropriate space $(i = 1, \cdots, p)$. The exact procedure is best described in an algorithmic framework, and we leave the details to a forthcoming paper on this subject.

The procedure outlined above shows that we can confine ourselves to the case that CON $(G; r, k, d)$ is full-dimensional, and we will do so in the following. There is another (technical) reason for this. If polyhedra are not full-dimensional, statements about non-redundancy of certain systems often become quite ugly due to the necessity to exclude equivalent inequalities. This is also true in our case. It is not difficult to derive the results for the lower dimensional cases from the results presented later. But the statements of these theorems are often rather complicated and we want to avoid unnecessary technicalities—there are enough technicalities in this paper anyway.

Before continuing let us remark that there is an easy way to improve upon the formulation of (2.1) by excluding a number of inequalities that are obviously redundant.

Given $G = (V, E)$ and $r, k, d \in \mathbf{Z}_+^{E_V}$, let us extend the functions r and d to functions operating on sets by setting

$$r(W) \equiv \max \{ r_{st} \mid s \in W, t \in V \setminus W \} \text{ for } W \subseteq V \tag{3.5}$$

and

(3.6) $d(Z,W) \equiv \max\{d_{st} \mid s \in W, t \in V\setminus(Z \cup W), k_{st} \geqq |Z|\}$ for $Z, W \subseteq V$.

We call a pair (Z, W), Z, $W \subseteq V$, *eligible* (with respect to k) if $Z \cap W = \emptyset$ and $|Z| = k_{st}$ for at least one pair of nodes with $s \in W$ and $t \in V\setminus(Z \cup W)$. Then CON $(G; r, k, d)$ is clearly contained in the solution set of the following system of equations and inequalities.

(3.7)

(i) $\sum_{i \in W} \sum_{j \in V\setminus W} x_{ij} \geqq r(W)$ for all $W \subseteq V$, $\emptyset \neq W \neq V$;

(ii) $\sum_{i \in W} \sum_{j \in V\setminus(W \cup Z)} x_{ij} \geqq d(Z, W)$ for all eligible pairs (Z, W) of subsets of V;

(iii) $x_{ij} \leqq 1$ for all $ij \in E\setminus \mathrm{ES}(G; r, k, d)$;

(iv) $x_{ij} = 1$ for all $ij \in \mathrm{ES}(G; r, k, d)$;

(v) $x_{ij} \geqq 0$ for all $ij \in E\setminus \mathrm{ES}(G; r, k, d)$ with $\mathrm{ES}(G; r, k, d) = \mathrm{ES}(G - e; r, k, d)$.

In the subsequent sections of this paper we will investigate in more detail these and other inequalities for special choices of r, k, and d.

4. Connectivity. We will now consider one of the easiest special cases of our network design problem (2.1). This case, however, will provide further insight and yield a new class of interesting inequalities.

Given a graph $G = (V, E)$ and $W \subseteq V$, the edge set

$$\delta(W) \equiv \{ij \in E \mid i \in W, j \in V\setminus W\}$$

is called the *cut* (induced by W). (We will write $\delta_G(W)$ to make clear—in case of possible ambiguities—with respect to which graph the cut induced by W is considered.) For W, $W' \in V$ with $W \cap W' = \emptyset$ we define $[W : W'] \equiv \{ij \in E \mid i \in W, j \in W'\}$. So $\delta(W) = [W : V\setminus W]$. For $W \subseteq V$ we set $E(W) \equiv \{ij \in E \mid i, j \in W\}$.

In this section we assume that the underlying graph $G = (V, E)$ is connected. As before, multiple edges are allowed, but loops are not. Let $\mathbf{1}$ be the vector (of appropriate dimension) with all components equal to 1. Set

(4.1) $$\mathrm{CON}(G) := \mathrm{CON}(G; \mathbf{1}, 0, 0).$$

In other words, CON (G) is the convex hull of all feasible solutions of the system

(4.2)

(i) $x(\delta(W)) \geqq 1$ for all $W \subseteq V, \emptyset \neq W \neq V$;

(ii) $0 \leqq x_e \leqq 1$ for all $e \in E$;

(iii) $x_e \in \{0, 1\}$ for all $e \in E$,

where, from now on, we use the symbol $x(F)$ to abbreviate the sum $\sum_{e \in F} x_e$. Another way to state (4.1) is

$$\mathrm{CON}(G) = \mathrm{conv}\{\chi^F \in \mathbf{R}^E \mid (V, F) \text{ is a connected subgraph of } G\}.$$

That is why we call CON (G) the *connected subgraph polytope of* G. It is easy (and well known how) to solve

(4.3) $$\min c^T x, x \in \mathrm{CON}(G).$$

This goes as follows. Let F be the set of edges e with nonpositive weight c_e. If (V, F) is connected, stop. Otherwise contract the components of (V, F) to single nodes and compute a minimum spanning tree T in the resulting graph. $T \cup F$ yields an optimum solution to (4.3). It is also well known that the solution set of the LP-relaxation of (4.2), i.e., the polyhedron defined by (i), (ii) of (4.2), is not integral in general, and that a complete linear description of CON (G) can be easily derived (see e.g., Cornuéjols, Fonlupt, and Naddef (1985)) from Edmonds' characterization of matroid polytopes—see Edmonds (1970, 1971). We give this transformation here for sake of completeness.

Recall that the bases of the graphic matroid on G are the spanning trees of G, that the bases of the cographic matroid are the complements of spanning trees, and that a set is independent in a matroid if it is contained in a basis. Let r (r^*, respectively) denote the rank function of the graphic (cographic, respectively) matroid on $G = (V, E)$. Then, for $F \subseteq E$, $r(F) = |V| - c_F$, where c_F is the number of components of (V, F), and

$$r^*(F) = |F| + r(E \backslash F) - r(E) = |F| - |V| + 1 + r(E \backslash F). \tag{4.4}$$

Let IND* (G) be the convex hull of the incidence vectors χ^F, where $F \subseteq E$ is independent in the cographic matroid of G, i.e.,

$$\text{IND}^*(G) = \text{conv}\{\chi^F \in \mathbf{R}^E \mid \exists \text{ spanning tree } T \text{ such that } F \subseteq E \backslash T\}. \tag{4.5}$$

The above definitions imply:

$$\text{CON}(G) = \{\mathbf{1} - y \in \mathbf{R}^E \mid y \in \text{IND}^*(G)\}, \tag{4.6}$$

$$\text{IND}^*(G) = \{\mathbf{1} - x \in \mathbf{R}^E \mid x \in \text{CON}(G)\}. \tag{4.7}$$

A subset $F \subseteq E$ is called *r^*-closed* if $r^*(F) < r^*(F \cup \{e\})$ for all $e \in E \backslash F$, and F is called *r^*-inseparable* if there is no partition F_1, F_2 of F such that $r^*(F) = r^*(F_1) + r^*(F_2)$. (A family $S_1, \cdots, S_m$ of subsets of a set S is called a *partition* of S if $S_i \neq \emptyset$, $i = 1, \cdots, m$; $S_i \cap S_j = \emptyset$, $1 \leqq i < j \leqq m$; and $S_1 \cup \cdots \cup S_m = S$.) Let $B(G) \subseteq E$ denote the set of bridges of G (a *bridge* is an edge that forms a cut). It follows from Edmonds' results on matroid polytopes that

$$\text{IND}^*(G) = \{y \in \mathbf{R}^E \mid \begin{array}{ll} y_e = 0 & \text{for all } e \in B(G); \\ y_e \geqq 0 & \text{for all } e \in E \backslash B(G); \\ y(F) \leqq r^*(F) & \text{for all } F \subseteq E, \text{with } F \\ & r^*\text{-closed and } r^*\text{-inseparable}\}. \end{array} \tag{4.8}$$

In fact, the linear description of IND* (G) given above is nonredundant. Using relation (4.6) and formula (4.4) we obtain the following nonredundant description of the connected subgraph polytope:

$$\text{CON}(G) = \{x \in \mathbf{R}^E \mid \begin{array}{ll} x_e = 1 & \text{for all } e \in B(G); \\ x_e \leqq 1 & \text{for all } e \in E \backslash B(G); \\ x(F) \geqq |V| - 1 - r(E \backslash F) & \text{for all } F \subseteq E \text{ with } F \\ & r^*\text{-closed and} \\ & r^*\text{-inseparable}\}. \end{array} \tag{4.9}$$

Observe that the bridges of a graph are exactly the $(G; \mathbf{1}, 0, 0)$-essential edges and recall that a connected graph is bridgeless if and only if it is 2-edge connected. It is a nice exercise to translate the matroid properties "r^*-closed" and "r^*-inseparable" into the language of graph theory. The result (ignoring the technical details coming up by considering bridges) is the following theorem.

THEOREM 4.10. *Let $G = (V, E)$ be a 2-edge connected graph. Then* CON (G) *is full-dimensional and*

(i) $\frac{1}{2}\sum_{i=1}^{p} x(\delta(V_i)) \geqq p - 1$ *for all partitions $V_1, \cdots, V_p$ of V, $p \geqq 2$, such that each subgraph $(V_i, E(V_i))$ is 2-edge connected and the graph obtained by contracting every V_i, $i = 1, \cdots, p$ to a single node is 2-node connected;*

(ii) $x_e \leqq 1$ *for all $e \in E$;*

(iii) $x_e \geqq 0$ *for all $e \in E$ such that $(G - e)$ is 2-edge connected*

is a complete and nonredundant linear characterization of CON (G). □

Since we can optimize any linear function over CON (G) in polynomial time, it follows from the ellipsoid method (see Grötschel, Lovász, and Schrijver (1981, 1988)) that the separation problem for the linear system (i)–(iii) of (4.10) can be solved in polynomial time. In fact, a specialization of Cunningham's algorithm for the separation problem for matroid polytopes (see Cunningham (1984)) yields a combinatorial separation algorithm for this system.

5. Steiner trees. Let $G = (V, E)$ be a connected graph, and let S, $|S| \geqq 2$, be a subset of the node set. S is called the set of *terminal* nodes, $V \backslash S$ is called the set of *Steiner nodes*. Define a vector $r^S \in \mathbf{Z}_+^{E_V}$ by setting $r_{st}^S \equiv 1$ for all $s, t \in S$, $s \neq t$ and $r_{st}^S = 0$ else, and let

$$\text{CON}(G;S) \equiv \text{CON}(G;r^S,0,0). \tag{5.1}$$

Then CON $(G; S)$ is the convex hull of all incidence vectors χ^F such that all nodes in S belong to the same component of (V, F). Another way to say this is that CON $(G; S)$ is the convex hull of all incidence vectors χ^F where F contains a Steiner tree of G (with $V \backslash S$ being the set of Steiner nodes). Thus for $c \in \mathbf{R}_+^E$ every optimum vertex solution of

$$\min c^T x, \qquad x \in \text{CON}(G;S) \tag{5.2}$$

yields an optimum Steiner tree. The LP-relaxation for (5.2) that is provided by (2.1), respectively, (3.7) and our special choice of r, k, and d has the following constraints:

(5.3) (i) $x(\delta(W)) \geqq 1$ for all $W \subseteq V$ such that $W \cap S \neq \varnothing$ and $S \backslash W \neq \varnothing$;

(ii) $0 \leqq x_e \leqq 1$ for all $e \in E$.

For $|S| = 2$, say $S = \{s, t\}$, the integral solutions of (5.3) are precisely the incidence vectors of edge sets $F \subseteq E$ that contain an $[s, t]$-path. In fact, it is easy to derive from any shortest path algorithm that the polyhedron defined by (5.3) is integral; thus in case $|S| = 2$, CON $(G; S) = \{x \in \mathbf{R}^E | x$ satisfies (5.3) (i) and (ii)$\}$ holds.

Let us call an edge $e \in E$ a *Steiner bridge* if $G - e$ contains no $[s, t]$-path for some nodes $s, t \in S$. We denote the *set of Steiner bridges* of a graph G by $B(G; S)$. It is easy to see that $B(G; S) = ES(G; r^S, 0, 0)$. For $S = V$ the Steiner bridges are just the bridges of G, i.e., $B(G; V) = B(G)$; for $|S| = 2$, say $S = \{s, t\}$, the Steiner bridges are the edges that are on every $[s, t]$-path. Such edges are called $[s, t]$-*bridges*. Let us denote the subgraph of G induced by the node set W by $G[W]$, i.e., $G[W] = (W, E(W))$. Using this terminology, a nonredundant characterization of the convex hull of the incidence vectors of edge sets containing an $[s, t]$-path can be derived easily.

THEOREM 5.4. *Let $G = (V, E)$ be a connected graph, let $s, t \in V$ be two different nodes and assume that G contains no $[s, t]$-bridge. Then the dimension of* CON $(G; \{s, t\})$ *is equal to $|E|$ and the following system is a complete and nonredundant characterization of* CON $(G; \{s, t\})$:

(i) $x(\delta(W)) \geqq 1$ *for all cuts $\delta(W)$ such that $s \in W$, $t \in V \setminus W$, and $G[W]$ and $G[V \setminus W]$ are connected*;

(ii) $x_e \leqq 1$ *for all $e \in E$*;

(iii) $x_e \geqq 0$ *for all $e \in E$ such that $G - e$ contains no $[s, t]$-bridge.*

The system (5.3) is in general not a complete description of CON $(G; S)$ if S is any set of terminal nodes with $|S| \geqq 3$ and G a general graph, not even for the other "extreme and simple" case where $S = V$, as Theorem (4.10) shows (note that CON (G) = CON $(G; V)$).

A natural way to generalize the system given in (4.10) to the Steiner tree problem is to consider the following system of inequalities:

(5.5) (i) $\frac{1}{2}\sum_{i=1}^{p} x(\delta(V_i)) \geqq p - 1$ for all partitions $V_1, \cdots, V_p$ of V, $p \geqq 2$, such that $|V_i \cap S| \geqq 1$ for $i = 1, \cdots, p$;

(ii) $0 \leqq x_e \leqq 1$ for all $e \in E$.

Clearly, all inequalities of the system (5.5) are valid for CON $(G; S)$; but—as observed by White, Farber, and Pulleyblank (1985)—they are not sufficient to describe CON $(G; S)$, not even for graphs as simple as series-parallel graphs.

System (5.5) seems, however, to be a reasonable LP-relaxation of the Steiner tree problem as the following result shows.

THEOREM 5.6. *Let $G = (V, E)$ be a connected graph, let $S \subseteq V$ be a set of terminal nodes and assume that G contains no Steiner bridge. Let $V_1, \cdots, V_p, p \geqq 2$, be a partition of V such that $V_i \cap S \neq \emptyset$ for $i = 1, \cdots, p$. Then*

$$\frac{1}{2}\sum_{i=1}^{p} x(\delta(V_i)) \geqq p-1$$

defines a facet of CON $(G; S)$ *if and only if*

(a) *$G[V_i]$ is connected for $i = 1, \cdots, p$*;

(b) *$G[V_i]$ contains no Steiner bridge with respect to the set $S_i := S \cap V_i$ of terminal nodes for $i = 1, \cdots, p$*;

(c) *the graph $\hat{G} = (\hat{V}, \hat{E})$ obtained from G by contracting each node set V_i to single node is 2-node connected.*

(Comment: If $|S_i| = 1$, no edge of $G[V_i]$ is a Steiner bridge.)

Proof. Suppose one of the graphs $G[V_i]$, say $G[V_1]$, is not connected. Let V_1' be the node set of a component of $G[V_1]$ such that $(V_1 \setminus V_1') \cap S \neq \emptyset$. Since G is connected there is a node set $V_j, j \in \{2, \cdots, p\}$, say V_2, such that V_1' and V_2 are connected by an edge. But then the inequality $\frac{1}{2}(x(\delta(V_1 \setminus V_1')) + x(\delta(V_2 \cup V_1')) + \sum_{i=3}^{p} x(\delta(V_i)) \geqq p - 1$ belongs to class (5.5) (ii) and its sum with $x_e \geqq 0$ for all $e \in [V_1' : V_2]$ is equal to $\frac{1}{2}\sum_{i=1}^{p} x(\delta(V_i)) \geqq p - 1$. So the latter inequality does not define a facet. We may thus assume that $G[V_i]$ is connected for all i.

Suppose $G[V_i]$, for some $i \in \{1, \cdots, p\}$, contains a Steiner bridge, say e, with respect to S_i. We will show that every $D \subseteq E$ with $\chi^D \in \mathrm{CON}(G; S)$ and satisfying $a^T x \equiv \frac{1}{2} \sum_{i=1}^{p} x(\delta(V_i)) \geqq p - 1$ with equality contains e. This implies that $a^T x \geqq p - 1$ does not define a facet. Suppose there exists $D \subseteq E \setminus \{e\}$ with $\chi^D \in \mathrm{CON}(G; S)$ and $a^T \chi^D = p - 1$. Since e is a Steiner bridge of $G[V_i]$, D must contain a path linking two Steiner nodes contained in different components of $G[V_i] - e$. Thus $D' \equiv D \cup E(V_i)$ satisfies $\chi^{D'} \in \mathrm{CON}(G; S)$ and $a^T \chi^{D'} = p - 1$ and contains a cycle C containing e and an edge, say f, of $\delta(V_i)$. But then $D'' \equiv (D' \setminus \{f\}) \cup \{e\}$ satisfies $\chi^{D''} \in \mathrm{CON}(G; S)$ and $a^T \chi^{D''} = p - 2$, a contradiction.

Suppose $\hat{G}$ is not 2-node connected. Let $\hat{V} = \{v_1, \cdots, v_p\}$, where v_i is the node obtained by contracting V_i, $i = 1, \cdots, p$. We may assume that v_1 is a cut-node such that $\{v_2, \cdots, v_c\}$ is the node set of one component of $\hat{G} - v_1$. Set $W_1 \equiv V_1 \cup \bigcup_{i=c+1}^{p} v_i$ and $W_2 \equiv V_1 \cup \bigcup_{i=2}^{p} v_i$. Then $\frac{1}{2} \sum_{i=1}^{p} x(\delta(V_i)) \geqq p - 1$ is the sum of the two valid inequalities $\frac{1}{2}(x(\delta(W_1)) + \sum_{i=2}^{c} x(\delta(V_i))) \geqq c - 1$ and $\frac{1}{2}(x(\delta(W_2)) + \sum_{i=c+1}^{p} x(\delta(V_i))) \geqq p - c$.

This shows that if one of the conditions (a), (b), (c) is not satisfied then the given inequality does not define a facet of $\mathrm{CON}(G; S)$.

Now let $a^T x \equiv \frac{1}{2} \sum_{i=1}^{p} x(\delta(V_i)) \geqq p - 1$ be an inequality of type (5.5) (ii) such that (a), (b), and (c) are satisfied. Let $b^T x = \beta$ be an equation such that $F_a \equiv \{x \in \mathrm{CON}(G; S) \mid a^T x = p - 1\} \subseteq F_b \equiv \{x \in \mathrm{CON}(G; S) \mid b^T x = \beta\}$ and such that F_b is a facet of $\mathrm{CON}(G; S)$.

Note that, by construction, every spanning tree T of $\hat{G}$ corresponds to a forest (also denoted by T) of G such that, for each subset B of $A \equiv \bigcup_{i=1}^{p} E(V_i)$, the incidence vector of $B \cup T$ satisfies $a^T x \geqq p - 1$ with equality.

We first prove that $b_e = 0$ for all $e \in A$. Let $e \in E(V_i)$ for some $i \in \{1, \cdots, p\}$. Choose two nodes $s \in S \cap V_i$ and $t \in S \cap (V \setminus V_i)$. Since G contains no Steiner bridge there exists an $[s, t]$-path P in G not containing e. Choose an edge $f \in P \cap \delta(V_i)$ and construct a spanning tree T of $\hat{G}$ containing f. Set $D \equiv T \cup A$ and $D_e \equiv D \setminus \{e\}$. Since $G[V_j]$ is connected for every j and since e is not a Steiner bridge of $G[V_i]$, by construction, $\chi^D, \chi^{D_e} \in \mathrm{CON}(G; S)$ and $a^T \chi^D = a^T \chi^{D_e} = p - 1$. Thus $b^T \chi^D = b^T \chi^{D_e}$ which implies $b_e = 0$.

Let e, f be different elements of $\hat{E}(=E \setminus A)$. Since $\hat{G}$ is 2-node connected there exists a cycle C of $\hat{G}$ containing e and f. Let T be a spanning tree of $\hat{G}$ containing $C \setminus \{e\}$ (but not f). Then $T' \equiv (T \setminus \{e\}) \cup \{f\}$ is also a spanning tree of $\hat{G}$. Set $D \equiv T \cup A$ and $D' \equiv T' \cup A$. Clearly $\chi^D, \chi^{D'} \in \mathrm{CON}(G; S)$ and $0 = b^T \chi^D - b^T \chi^{D'} = b_e - b_f$. This implies that $b^T x$ is a multiple of $a^T x$ which proves that $a^T x \geqq p - 1$ defines a facet of $\mathrm{CON}(G; S)$. □

Theorems 3.2, 3.3, and 5.6 immediately give the following result.

COROLLARY 5.7. *Let $G = (V, E)$ be a connected graph, $S \subseteq V$ a set of at least two terminal nodes and assume that G contains no Steiner bridge. Then* $\dim(\mathrm{CON}(G; S)) = |E|$ *and the following system is a nonredundant system of facet-defining inequalities for* $\mathrm{CON}(G; S)$:

(i) $\displaystyle \frac{1}{2} \sum_{i=1}^{p} x(\delta(V_i)) \geqq p - 1$ *for all partitions $V_1, \cdots, V_p$ of V, $p \geqq 2$, such that $\hat{G}$ is 2-node connected and, for $i = 1, \cdots, p$, $V_i \cap S \neq \emptyset$ and $G[V_i]$ is connected and contains no Steiner bridge with respect to $V_i \cap S$;*

(ii) $x_e \leqq 1$ *for all $e \in E$;*

(iii) $x_e \geqq 0$ *for all $e \in E$ such that $G - e$ contains no Steiner bridge.*

Note that the nonredundancy parts of Theorems 4.10 and 5.4 follow directly from Corollary (5.7).

Based on the paper of Prodon, Liebling, and Gröflin (1985), Prodon (1985) has generalized the inequality system (5.5) (i) for CON $(G; S)$ to the system (5.9) defined below. We give a polyhedral proof of the validity of these inequalities.

PROPOSITION 5.8. *Let $G = (V, E)$ be a graph and $S \subseteq V$, $|S| \geqq 2$, a set of terminal nodes. Let $\mathcal{F}$ be a set of subsets of V such that*

(a) $|\mathcal{F}| \geqq 1$,

(b) $U \cap S \neq \emptyset$ *for all* $U \in \mathcal{F}$,

(c) $(V \setminus \bigcup_{U \in \mathcal{F}} U) \cap S \neq \emptyset$.

For each edge $e = uv \in E$, set

$$\Delta(\mathcal{F}; e, u, v) \equiv |\{U \in \mathcal{F} \mid u \in U, v \notin U\}|,$$

$$a_e(\mathcal{F}) \equiv \max\{\Delta(\mathcal{F}; e, u, v), \Delta(\mathcal{F}; e, v, u)\},$$

and define

$$a(\mathcal{F}) \equiv (a_e(\mathcal{F}))_{e \in E} \in \mathbf{R}^E.$$

Then

$$a(\mathcal{F})^T x \geqq |\mathcal{F}| \tag{5.9}$$

is valid with respect to CON $(G; S)$.

Proof. We prove the validity of (5.9) by induction on $|\mathcal{F}|$. For $|\mathcal{F}| = 1$, say $\mathcal{F} = \{W\}$, (5.9) is nothing but the (valid) cut inequality $x(\delta(W)) \geqq 1$ already considered in (5.3) (i).

We now assume that (5.9) is valid for CON $(G; S)$ for all set systems satisfying (a), (b), (c) with at most $p \geqq 1$ elements. Let $\mathcal{F} \subseteq 2^V$ be a set system satisfying (a), (b), (c) with $p + 1$ elements. For ease of notation, let us set $\bar{V} \equiv V \setminus (\bigcup_{u \in \mathcal{F}} U)$.

Let I be the set of all (unordered) pairs $\{U, W\}$ with $U, W \in \mathcal{F}$, such that there is an edge $e \in E$ with one endnode in U (or in W) and the other endnode in $W \setminus U$ (or in $U \setminus W$). Moreover, let J be the set of nodesets $U \in \mathcal{F}$ such that there exists an edge $e \in E$ with one endnode in U and the other endnode in $\bar{V}$.

We define new set systems as follows. For $\{U, W\} \in I$, set

$$\mathcal{F}_{UW} \equiv (\mathcal{F} \setminus \{U, W\}) \cup \{U \cup W\},$$

and for $U \in J$, set $\mathcal{F}_U \equiv \mathcal{F} \setminus \{U\}$. Clearly, each of the systems $\mathcal{F}_{UW}$ and $\mathcal{F}_U$ has cardinality p and satisfies (a), (b), and (c). Let

$$b^T x \equiv \sum_{\{U,W\} \in I} a(\mathcal{F}_{UW})^T x + \sum_{U \in J} a(\mathcal{F}_U)^T x.$$

By induction hypothesis, the inequalities of type (5.9) associated with the systems $\mathcal{F}_{UW}$ and the systems $\mathcal{F}_U$ are valid for CON $(G; S)$. Thus $b^T x \geqq (|I| + |J|)p$ is also valid.

We will now prove (componentwise) that $(|I| + |J| - 1)a(\mathcal{F})^T x \geqq b^T x$. From this observation we can conclude that $a(\mathcal{F})^T x \geqq (|I| + |J|)/(|I| + |J| - 1)p$, and thus validity of $a(\mathcal{F})^T x \geqq |\mathcal{F}|$ follows by rounding up the right-hand side.

By definition, for every edge $e \in E$, $a_e(\mathcal{F}) \geqq a_e(\mathcal{F}_{UW}) \geqq a_e(\mathcal{F}) - 1$ for all $\{U, W\} \in I$, and $a_e(\mathcal{F}) \geqq a_e(\mathcal{F}_U) \geqq a_e(\mathcal{F}) - 1$ for all $U \in J$; moreover, $a(\mathcal{F}_{UW}) \geqq$

0 for all $\{U, W\} \in I$ and $a(\mathscr{F}_U) \geqq 0$ for all $U \in J$. Thus, for every edge $e \in E$ with $a_e(\mathscr{F}) = 0$, we have $b_e = 0$.

Let $e = uv \in E$ be an edge with $a_e(\mathscr{F}) > 0$.

Case 1. $u, v \notin \bar{V}$.

Case 1.1. $\Delta(\mathscr{F}; e, u, v) > \Delta(\mathscr{F}; e, v, u)$. Set $\mathscr{F}' \equiv \{U' \in \mathscr{F} \mid u \in U', v \notin U'\}$. By definition there are at least $a_e(\mathscr{F})$ sets in $\mathscr{F}'$, and moreover, since $v \notin \bar{V}$ there is at least one set, say W', in $\mathscr{F} \setminus \mathscr{F}'$ with $v \in W'$. Set $I' \equiv \{\{U', W'\} \mid U' \in \mathscr{F}'\}$. Then $I' \subseteq I$ and $a_e(\mathscr{F}_{U'W'}) = a_e(\mathscr{F}) - 1$ for all $\{U', W'\} \in I'$. Thus

$$b_e = \sum_{\{U,W\} \in I} a_e(\mathscr{F}_{UW}) + \sum_{U \in J} a_e(\mathscr{F}_U) \leqq (|I| + |J|)a_e(\mathscr{F}) - |I'|$$

$$\leqq (|I| + |J| - 1)a_e(\mathscr{F}).$$

The case $\Delta(\mathscr{F}; e, v, u) > \Delta(\mathscr{F}; e, u, v)$ follows by symmetry.

Case 1.2. $\Delta(\mathscr{F}; e, v, u) = \Delta(\mathscr{F}; e, u, v)$. Let $\mathscr{F}' \equiv \{U' \in \mathscr{F} \mid u \in U', v \notin U'\}$ and $\mathscr{F}'' \equiv \{U'' \in \mathscr{F} \mid u \notin U'', v \in U''\}$. By assumption $|\mathscr{F}'| = |\mathscr{F}''| = a_e(\mathscr{F})$. Set $I' \equiv \{\{U', U''\} \mid U' \in \mathscr{F}', U'' \in \mathscr{F}''\}$. Then $|I'| = a_e(\mathscr{F})^2$ and $a_e(\mathscr{F}_{U'U''}) = a_e(\mathscr{F}) - 1$ for all $\{U', U''\} \in I'$. This implies as before $b_e \leqq (|I| + |J|)a_e(\mathscr{F}) - |I'| \leqq (|I| + |J| - 1)a_e(\mathscr{F})$.

Case 2. $e \in \delta(\bar{V})$, say $u \in V \setminus \bar{V}$ and $v \in \bar{V}$. Let $\mathscr{F}' \equiv \{U \in \mathscr{F} \mid u \in U\}$. Then by definition $a_e(\mathscr{F}) = \Delta(\mathscr{F}; e, u, v) = |\mathscr{F}'|$ and $\mathscr{F}' \subseteq J$. Obviously, $a_e(\mathscr{F}_U) = a_e(\mathscr{F}) - 1$ for all $U \in \mathscr{F}'$, and hence

$$b_e \leqq (|I| + |J|)a_e(\mathscr{F}) - |\mathscr{F}'| \leqq (|I| + |J| - 1)a_e(\mathscr{F}).$$

This finishes the proof. □

It is easy to find some necessary conditions for an inequality (5.9) to define a facet of CON (G). But it is currently unknown which of the inequalities (5.9) defines facets of CON $(G; S)$.

Given a graph $G = (V, E)$, a node set $S \subseteq V$, and a vector $y \in \mathbf{R}^E$ (we may assume $0 \leqq y \leqq \mathbf{1}$), it is easy to solve the separation problem for y and the inequality system (5.3) (i) by computing a Gomory–Hu tree for the graph $G = (V, E)$ with the values y_e, $e \in E$, considered as edge capacities. (The Gomory–Hu method for finding a minimum capacity cut is, for instance, described in Hu (1969) or Grötschel, Lovász, and Schrijver (1988).) The Gomory–Hu tree contains an edge with capacity smaller than one whose removal separates two nodes in S if and only if y violates at least one of the inequalities (5.3) (i). This implies (see, e.g., Grötschel, Lovász, and Schrijver (1988)) that the LP-relaxation of (5.2) coming from (5.3) can be solved in polynomial time.

The Gomory–Hu tree can be used to find some violated inequalities of type (5.5) (i) heuristically. But a polynomial time separation procedure for the system (5.5) is not known. The same holds for Prodon's system of inequalities defined in Proposition 5.8.

6. Edge connectivity. By setting $r_{st} \equiv k$ (k a positive integer), and $d_{st} \equiv k_{st} \equiv 0$ for all $s, t \in V$, $s \notin t$ we obtain the following special case of (2.1), respectively (3.7):

(6.1) $\min c^T x$

$$\begin{array}{rll} \text{(i)} & x(\delta(W)) \geqq k & \text{for all } W \subseteq V, \varnothing \neq W \neq V; \\ \text{(ii)} & 0 \leqq x_e \leqq 1 & \text{for all } e \in E; \\ \text{(iii)} & x_e \in \{0, 1\} & \text{for all } e \in E. \end{array}$$

The feasible solutions of (6.1) are exactly the incidence vectors of all edge sets $C \subseteq E$ such that (V, C) is k-edge connected (i.e., every pair of nodes of G is linked by k edge disjoint paths). So (6.1) is asking for a minimum cost spanning k-edge connected subgraph of G. To simplify notation in the following, we will just speak of a k-edge connected edge set C and mean that (V, C) is a (spanning) k-edge connected subgraph of G. Let us set

$$\text{ECON}\,(G;k) \equiv \text{conv}\,\{\chi^C \in \mathbf{R}^E \mid (V,C) \text{ is } k\text{-edge connected}\}.$$

ECON $(G; k)$ is called the *polytope of k-edge connected subgraphs of G*. Clearly, ECON $(G; k)$ = CON $(G; k\mathbf{1}, 0, 0)$, and for $k = 1$, we have ECON $(G; 1)$ = CON (G)—see § 4. The dominant of this polytope, i.e., the polyhedron ECON $(G; k) + \mathbf{R}^E_+$, has been considered in Cornuéjols, Fonlupt, and Naddef (1985) who showed, among other results, that ECON $(G; 2) + \mathbf{R}^E_+ = \{x \in \mathbf{R}^E \mid x \geqq 0,\ x(\delta(W)) \geqq 2$ for all $W \subseteq V,\ \varnothing \neq W \neq V\}$ if G is series-parallel.

We will now give a technical characterization of those inequalities of type (6.1) (i) that define facets of ECON $(G; k)$. To do this we introduce further notation. An edge e of a graph $G = (V, E)$ is called *k-edge-essential* if $G - e$ is not k-edge connected. So the k-edge-essential edges of G are exactly the edges that are $(G; k\mathbf{1}, 0, 0)$-essential. By Theorem (3.2), ECON $(G; k)$ has dimension $|E|$ if and only if G is k-edge connected and contains no k-edge-essential edge. By Menger's theorem the latter condition is equivalent to G and is $(k + 1)$-edge connected.

THEOREM 6.2. *Let $G = (V, E)$ be a $(k + 1)$-edge connected graph, $k \geqq 1$. Then, for $W \subseteq V$, $\varnothing \neq W \neq V$,*

$$x(\delta(W)) \geqq k$$

defines a facet of ECON $(G; k)$ *if and only if*

(a) *for each edge $e \in E \setminus \delta(W)$, there exists a set $C \subseteq \delta(W)$ such that*
  (a_1) $|C| = k$, *and*
  (a_2) $C \cup (E \setminus (\delta(W) \cup \{e\}))$ *is k-edge connected*; *and*
(b) *there exist edge sets $C_1, \cdots, C_s \subseteq \delta(W)$, where $s = |\delta(W)|$, such that*
  (b_1) $|C_i| = k,\ i = 1, \cdots, s$,
  (b_2) $C_i \cup (E \setminus (\delta(W))$ *is k-edge connected, and*
  (b_3) *the $s \times s$-matrix M whose columns are the incidence vectors $\chi^{C_i} \in \mathbf{R}^{\delta(W)}$ is nonsingular.*

Proof. Suppose (a) and (b) are satisfied. Set $a^T x \equiv x(\delta(W))$ and let $b^T x \geqq \beta$ define a facet F_b of ECON $(G; k)$ that contains the face $F_a \equiv \{x \in \text{ECON}\,(G; k) \mid a^T x = k\}$.

Let $e \in E \setminus \delta(W)$ and let $C_e \equiv C \cup (E \setminus (\delta(W) \cup \{e\}))$ be the k-edge connected subset of E existing by (a). Then $D \equiv C_e \cup \{e\}$ is also k-edge connected and, by (a_1) we have $a^T \chi^{C_e} = a^T \chi^D = k$. Thus, since $F_a \subseteq F_b$, $b^T \chi^{C_e} = b^T \chi^D$ holds. This implies $b_e = 0$. It follows that $b_e = 0$ for all $e \in E \setminus \delta(W)$.

The incidence vectors of the k-edge connected sets $D_i \equiv C_i \cup (E \setminus \delta(W))$, $i = 1, \cdots, s$, satisfy $a^T \chi^{D_i} = k$ and hence $b^T \chi^{D_i} = \beta$. Consider the equation $y^T M = \beta \mathbf{1}^T$. Clearly, the vectors $(\beta/k)\mathbf{1}^T$ and $\bar{b}^T$ in $\mathbf{R}^{\delta(W)}$ (where $\bar{b}$ is the vector obtained from b by deleting the components $E \setminus \delta(W)$) are solutions of this equation. Since M is nonsingular, $\bar{b} = (\beta/k)\mathbf{1}$ has to hold. This implies that $a = (k/\beta)^b$, and thus, $a^T x \geqq k$ defines a facet of ECON $(G; k)$.

If (a) does not hold then there is an edge $e \in E \setminus \delta(W)$ such that, for no subset C of $\delta(W)$ with $|C| = k$, the set $C \cup (E \setminus (\delta(W) \cup \{e\}))$ is k-edge connected. Hence either no k-edge connected edge set satisfies $x(\delta(W)) \geqq k$ with equality (and thus this inequality does not define a facet) or every k-edge connected subset $D \subseteq E$ with $|D \cap \delta(W)| = k$ contains e. Therefore $\{x \in \text{ECON}\,(G; k) \mid x(\delta(W)) = k\} \subseteq \{x \in \text{ECON}\,(G; k) \mid x_e = 1\}$ and $x(\delta(W)) \geqq k$ does not define a facet.

If (b) does not hold, then the set of vertices contained in the face

$$F \equiv \{x \in \mathrm{ECON}\,(G;k) \mid x(\delta(W)) = k\}$$

is not linearly independent. Hence, as ECON $(G; k)$ is full-dimensional, F is not a facet of ECON $(G; k)$. $\square$

Theorem 6.2 is merely of technical interest and does not provide insight into the graph theoretical properties that make an inequality $x(\delta(W)) \geqq k$ a facet-defining one. In fact, in contrast to our earlier belief, there seems no easy way to use connectivity properties only to decide whether or not such an inequality defines a facet of ECON $(G; k)$. This will become clear by the following observations.

PROPOSITION 6.3. *Let $G = (V, E)$ be $(k+1)$-edge connected, $k \geqq 2$, and $W \subseteq V$ with $\varnothing \neq W \neq V$ such that $G[W]$ and $G[V \setminus W]$ are k-edge connected. Then $x(\delta(W)) \geqq k$ defines a facet of* ECON $(G; k)$.

Proof. We first show that (a) of Theorem 6.2 is satisfied. Let $e = uv \in E \setminus \delta(W)$, say $e \in E(W)$. If $G' \equiv G[W] - e$ is k-edge connected then $C \cup (E \setminus (\delta(W) \cup \{e\}))$ is k-edge connected for every set $C \subseteq \delta(W)$ with $|C| = k$. So let us assume that $\lambda(G') = k - 1$. This implies that G' contains a cut of cardinality $k - 1$ that separates u and v. Among all node sets $W_u, W_v \subseteq W$ with $W_u \cap W_v = \varnothing$ such that $u \in W_u$, $v \in W_v$, $e \in \delta(W_u)$, $e \in \delta(W_v)$, $|\delta_{G'}(W_u)| = |\delta_{G'}(W_v)| = k - 1$ (note that such sets exist) we choose W_u and W_v such that W_u and W_v have cardinality as small as possible. Since G is $(k+1)$-edge connected there exists an edge $f \in E$ with one endnode in W_u and the other in $V \setminus W$, and one edge $g \in E$ with one endnode in W_v and the other in $V \setminus W$. Let C be any subset of $\delta(W)$ with $|C| = k$ and $f, g \in C$. (Here $k \geqq 2$ is needed.) We claim that $D \equiv C \cup (E \setminus (\delta(W) \cup \{e\}))$ is k-edge connected. Suppose not; then there must be a cut $\delta(Z)$ in $G'' = (V, D)$ with $k - 1$ or fewer edges. It follows from our assumptions that $\delta_{G''}(Z)$ does not separate any two nodes of $V \setminus W$, that it must separate u and v (so we may assume that $u \in Z$) and that $|\delta_{G'}(Z)| = k - 1$. Since the cut cardinality function $|\delta_{G'}|$ is submodular on the subsets of nodes of G' we obtain $2k - 2 \geqq |\delta_{G'}(W_u)| + |\delta_{G'}(Z)| \geqq |\delta_{G'}(W_u \cap Z)| + |\delta_{G'}(W \cup Z)|$. As $|\delta_{G'}(X)| \geqq k - 1$ for all cuts in G' we can conclude that $|\delta_{G'}(W_u \cap Z)| = k - 1$, and thus by the choice of W_u, we have $W_u = W_u \cap Z$. But then $f \in \delta_{G''}(Z)$ and thus $|\delta_{G''}(Z)| \geqq k$, a contradiction. This proves that (a) of (6.2) is satisfied.

For every $C \subseteq \delta(W)$ with $|C| = k$, $C \cup (E \setminus \delta(W))$ is k-edge connected and the matrix whose columns are the incidence vectors of all possible k-element subsets of $\delta(W)$ has obviously full row rank. Thus a matrix M as required by (b) of (6.2) exists. Hence the theorem follows from (6.2). $\square$

The next example shows that the connectivity requirements on $G[W]$ and $G[V \setminus W]$ cannot be weakened. They are, in a sense, best possible.

Example 6.4. Let $k \geqq 1$ and $G_1 = (V_1, E_1)$ be a minimal $(k-1)$-edge connected graph (i.e., G_1 is $(k-1)$-edge connected and each edge is $(k-1)$-edge essential) with at least $k+1$ nodes of degree $k - 1$. (Such graphs exist for all large enough orders.) Let G be the graph obtained from the disjoint union of G_1 and the complete graph $K_{k+1} = (V_2, E_2)$ of order $k + 1$ by adding all edges that link a node in G_1 to a node in K_{k+1}. G is clearly $(k+1)$-edge connected. The inequality $x(\delta(V_1)) \geqq k$ does not define a facet of ECON $(G; k)$, since there is no set $C \subseteq \delta(V_1)$ with $|C| = k$ such that $C \cup E_1 \cup E_2$ is k-edge connected (because at least one node in $(V_1 \cup V_2, C \cup E_1 \cup E_2)$ has degree $k - 1$).

On the other hand a cut inequality may define a facet of ECON $(G; k)$ even if $G[W]$ and $G[V \setminus W]$ are not highly connected as the next example shows.

Example 6.5. Consider four complete graphs on node sets A, B, C, D with $k \geqq 4$ nodes. Link all nodes in $A \cup B$ to all nodes in $C \cup D$, add a set of $\lceil k/2 \rceil + 1$ node disjoint edges with one endnode in A and one in B, and add a set of $\lceil k/2 \rceil + 1$ node disjoint edges with one endnode in C and the other in D. Let $G = (V, E)$ be the graph obtained this way. G is $(2k + \lceil k/2 \rceil + 1)$-edge connected. Let $W \equiv A \cup B$. Then the edge (and node) connectivity of $G[W]$ and $G[V \backslash W]$ is $\lceil k/2 \rceil + 1$. Using Theorem (6.2) it is easy to show that $x(\delta(W)) \geqq k$ defines a facet of ECON $(G; k)$.

The above example is an extreme case. The next observation shows that the connectivity of $G[W]$ and $G[V\backslash W]$ cannot be smaller than $\lceil k/2 \rceil + 1$ if the associated inequality defines a facet.

PROPOSITION 6.6. *Let $k \geqq 2$, $G = (V, E)$ be a $(k + 1)$-edge connected graph and $W \subseteq V$, $\emptyset \neq W \neq V$, such that the edge connectivity of $G[W]$ or $G[V \backslash W]$ is not larger than $\lceil k \backslash 2 \rceil$. Then $x(\delta(W))$ does not define a facet of* ECON $(G; k)$.

Proof. We may assume without loss of generality that $G[W]$ is $\lceil k/2 \rceil$-edge connected but not $(\lceil k/2 \rceil + 1)$-edge connected. By Menger's theorem $G[W]$ contains an edge e such that $G[W] - e$ is not $\lceil k/2 \rceil$-edge connected. It is easy to see that each k-edge connected edge set $F \subseteq E$ with $\chi^F(\delta(W)) = k$ contains edge e. Hence by Theorem 6.2, $x(\delta(W)) \geqq k$ does not define a facet of ECON $(G; k)$. □

Propositions 6.3 and 6.6 yield the following result for the cases $k = 2, 3$ which are of particular practical interest.

COROLLARY 6.7. *Let $k \in \{2, 3\}$ and let $G = (V, E)$ be a $(k + 1)$-edge connected graph. Let $W \subseteq V$, $\emptyset \neq W \neq V$. Then $x(\delta(W)) \geqq k$ defines a facet of* ECON $(G; k)$ *if and only if $G[W]$ and $G[V \backslash W]$ are k-edge connected.*

Recall that Theorem 4.10 (or Corollary 5.7) yields that, for a 2-edge connected graph G and for a node set $W \subseteq V$, $\emptyset \neq W \neq V$, $x(\delta(W)) \geqq 1$ defines a facet of ECON $(G; 1)$ if and only if $G[W]$ and $G[V\backslash W]$ are 2-edge connected.

Given $G = (V, E)$ and $y \in \mathbf{R}^E$, $0 \leqq y \leqq \mathbf{1}$, the separation problem for y and (6.1) (i) can be solved in polynomial time by computing a cut $\delta(W^*)$, $\emptyset \neq W^* \neq V$, of minimum capacity $y(\delta(W^*))$—for instance by the Gomory–Hu method. If $y(\delta(W^*)) < k$ the vector y violates $x(\delta(W^*)) \geqq k$, otherwise all inequalities (6.1) (i) are satisfied by y. Hence the LP-relaxation of the minimum cost k-edge connected subgraph problem following from (6.1) (by dropping the integrality constraints (iii)) can be solved in polynomial time.

7. Node connectivity. Parallel edges do not play a role in node connectivity questions. Thus we assume throughout this section that all graphs $G = (V, E)$ considered are simple. Setting in (2.1) $d_{st} \equiv 1$, $r_{st} \equiv k$, and $k_{st} \equiv k - 1$ ($k \in \mathbf{Z}$, $1 \leqq k \leqq |V| - 1$) for all $s, t \in V$, we obtain the following integer linear program:

(7.1)
$$\begin{array}{lll} & \min c^T x & \\ \text{(i)} & x(\delta(W)) \geqq k & \text{for all } W \subseteq V; \\ \text{(ii)} & x(\delta_{G-Z}(W)) \geqq 1 & \text{for all node sets } Z \subseteq V \text{ with} \\ & & |Z| = k-1 \text{ and all node sets} \\ & & W \subseteq V\backslash Z,\ \emptyset \neq W \neq V\backslash Z; \\ \text{(iii)} & 0 \leqq x_e \leqq 1 & \text{for all } e \in E; \\ \text{(iv)} & x_e \in \{0, 1\} & \text{for all } e \in E. \end{array}$$

Clearly, every optimum solution of (7.1) is a minimum-cost spanning k-node connected subgraph of G. The polytope

$$\text{NCON}\,(G;k) \equiv \text{conv}\,\{\chi^C \in \mathbf{R}^E \mid (V, C) \text{ is } k\text{-node connected}\}$$

is called the *polytope of k-node connected* (or just *k-connected*) *subgraphs* of G. For $k = 1$ this polytope coincides with ECON $(G; 1)$ and CON (G). Note that NCON $(G; k) \subseteq$

ECON $(G; k)$. Using the observations of § 4 we can immediately strengthen (7.1) to the following ILP:

(7.2)
$$\begin{array}{lll} & \min c^T x & \\ \text{(i)} & x(\delta(W)) \geqq k & \text{for all } W \subseteq V, \emptyset \neq W \neq V; \\ \text{(ii)} & \frac{1}{2}\sum_{i=1}^{p} x(\delta_{G-Z}(V_i)) \geqq p-1 & \text{for all } Z \subseteq V \text{ with } |Z| = k-1 \\ & & \text{and all nontrivial partitions} \\ & & V_1, \cdots, V_p \text{ of } V \backslash Z, p \geqq 2; \\ \text{(iii)} & 0 \leqq x_e \leqq 1 & \text{for all } e \in E; \\ \text{(iv)} & x_e \in \{0, 1\} & \text{for all } e \in E. \end{array}$$

It follows from our remarks in § 6 that the separation problem for (7.2) (i) and (iii) can be solved in polynomial time. Our remarks in § 4 imply that the separation problem for (7.2) (ii) and (iii) can be solved in polynomial time for every fixed node set Z. Thus, for k fixed, the separation problem for (ii) and hence the LP-relaxation of (7.2) are solvable in polynomial time.

We will now investigate NCON $(G; k)$ and find out which of the inequalities of (7.2) induce facets of the polytope of k-node connected subgraphs.

We will call an edge e of a graph $G = (V, E)$ *k-node essential* if $G - e$ is not k-node connected. The k-node-essential edges of G are thus precisely the $(G; k\mathbf{1}, (k-1)\mathbf{1}, \mathbf{1})$-essential edges of G. By Theorem 3.2, NCON $(G; k)$ has dimension $|E|$ if and only if G is k-node connected and contains no k-node-essential edge. In particular, NCON $(G; k)$ has dimension $|E|$ if G is $(k+1)$-node connected.

The results of § 6 concerning the cut inequalities $x(\delta(W)) \geqq k$ for ECON $(G; k)$ carry over to the case of node connectivity. Only minor modifications in the proofs and examples have to be made. We thus only summarize these observations and give no proofs. The main proof technique is an appropriate modification of Theorem 6.2.

THEOREM 7.3. (a) *Let $G = (V, E)$ be a k-node connected graph, $k \geqq 2$, without k-node-essential edge and let $W \subseteq V$, $\emptyset \neq W \neq V$, such that $G[W]$ and $G[V \backslash W]$ are k-node connected. Then $x(\delta(W)) \geqq k$ defines a facet of* NCON $(G; k)$, *cf.* (6.3).

(b) *If $x(\delta(W)) \geqq k$ defines a facet of* NCON $(G; k)$, *where $G = (V, E)$ is a k-node connected graph, $k \geqq 2$, without k-node-essential edge and $W \subseteq V$, $\emptyset \neq W \neq V$, then, for each $e \in E(W)$ and for each $f \in E(V \backslash W)$, $\kappa(G([W] - \{e\})$ and $\kappa(G[V \backslash W] - \{f\})$ are at least $\lceil k/2 \rceil$, cf.* (6.6).

(c) *For every $k \geqq 1$, there are $(k+1)$-node connected graphs $G = (V, E)$ and sets $W \subseteq V$ such that $G[W]$ is $(k-1)$-node connected, $G[V \backslash W]$ is $(k+1)$-node connected and such that $x(\delta(W)) \geqq k$ does not define a facet of* NCON $(G; k)$, *cf.* (6.4).

(d) *For every $k \geqq 4$, there are $(k+1)$-node connected graphs $G = (V, E)$ and sets $W \subseteq V$ such that $\kappa(G[W])$ and $\kappa(G[V \backslash W])$ are equal to $\lceil k/2 \rceil + 1$ and such that $x(\delta(W)) \geqq k$ defines a facet of* NCON $(G; k)$, *cf.* (6.5).

We will now give a technical characterization of those inequalities (7.2) (ii) that define facets of NCON $(G; k)$.

THEOREM 7.4. *Let $k \geqq 1$ and let $G = (V, E)$ be a k-node connected graph without k-node-essential edge. Let $Z \subseteq V$ with $|Z| = k-1$ and let $V_1, \cdots, V_p$, $p \geqq 2$, be a partition of $V \backslash Z$. Let $E' \equiv E(V_1) \cup \cdots \cup E(V_p) \cup E(Z) \cup \delta(Z)$ and let $\hat{G} = (\hat{V}, \hat{E})$ be the graph obtained from G by deleting Z and contracting $V_1, \cdots, V_p$. Then*

$$\frac{1}{2}\sum_{i=1}^{p} x(\delta_{G-Z}(V_i)) = x(E \backslash E') = x(\bar{E}) \geqq p-1$$

defines a facet of NCON $(G; k)$ *if and only if*

(a) *for each edge* $e \in E'$ *there exists a spanning tree* T *in* $\hat{G}$ *such that* $T \cup E' \setminus \{e\}$ *is k-node connected*; *and*

(b) *there exists a set* $T_1, \cdots, T_s$ *of spanning trees in* $\hat{G}$, *where* $s = |\hat{E}|$, *such that*:

(b_1) $T_i \cup E'$ *is k-node connected, and*

(b_2) *the* $s \times s$*-matrix* M *whose columns are the incidence vectors* $\chi^{T_i} \in \mathbf{R}^{\hat{E}}$ *is nonsingular*.

Proof. Let $Z \subseteq V$ with $|Z| = k - 1$ and a partition $V_1, \cdots, V_p$ of $V \setminus Z$, $p \geqq 2$, be given. Set $a^T x \equiv \frac{1}{2} \sum_{i=1}^{p} x(\delta_{G-Z}(V_i))$, $F_a \equiv \{x \in \text{NCON}(G; k) \mid a^T x = p - 1\}$, and let us call an edge set $C \subseteq E$ *tight* if (V, C) is k-node connected and $a^T \chi^C = p - 1$, i.e., if $\chi^C \in F_a$. Note that $E = E' \cup \hat{E}$, and since G is k-node connected and without k-node-essential edge, $\hat{G}$ is 2-edge connected.

Suppose (a) and (b) hold and that $b^T x \geqq \beta$ defines a facet F_b of NCON $(G; k)$ containing F_a. Condition (a) implies that, for each edge $e \in E'$, there exists a spanning tree T in $\hat{G}$ such that $C_e \equiv T \cup E' \setminus \{e\}$ is k-node connected. Since $|C_e \cap \bigcup_{i=1}^{p} \delta_{G-Z}(V_i)| = p - 1$ we can conclude that C_e is tight, hence, since $F_a \subseteq F_b$, $b^T \chi^{C_e} = \beta$ holds. But, for $C'_e \equiv C_e \cup \{e\}$, we also have $a^T \chi^{C'_e} = p - 1$ and thus $b^T \chi^{C'_e} = \beta$. This implies $b^T \chi^{C_e} = b^T \chi^{C'_e}$ and hence $b_e = 0$ for each $e \in E'$.

$D_i \equiv T_i \cup E'$ is tight for $i = 1, \cdots, s$ by (b). The equation $y^T M = \beta \mathbf{1}^T$ is solved by the vectors $(\beta/(p-1))\mathbf{1}^T$ and $\bar{b}^T$ of $\mathbf{R}^{\hat{E}}$ (where $\bar{b}$ is the vector obtained from b by deleting the components E'). Since M is nonsingular, we have $\bar{b}^T \equiv (\beta/(p-1))\mathbf{1}^T$. Clearly $\beta/(p-1) \neq 0$, and thus we can conclude that $a = ((p-1)/\beta) b$ which implies that F_a defines a facet of NCON $(G; k)$.

If (a) does not hold then there is either no tight set at all (and hence $a^T x \geqq p - 1$ does not define a facet) or there is an edge $e \in E'$ such that each tight set $C \subseteq E$ contains e. This implies that $F_a \subseteq \{x \in \text{NCON}(G; k) \mid x_e = 1\}$ and hence F_a does not define a facet. If (b) does not hold then the set of vertices contained in the face F_a does not span a hyperplane of $\mathbf{R}^E$, and thus, as NCON $(G; k)$ has dimension $|E|$, F_a is not a facet of NCON $(G; k)$. $\square$

As in the case of Theorem 6.2 we see no way to translate conditions (a) and (b) of Theorem 7.4 equivalently into nice graph theoretical properties, in particular, into connectivity requirements. An easy consequence of Theorem 7.4 is the following.

THEOREM 7.5. *Let* $G = (V, E)$ *be a k-node connected graph*, $k \geqq 1$, *without k-node-essential edge. Let* $Z \subseteq V$ *with* $|Z| = k - 1$, *and let* $V_1, \cdots, V_p$, $p \geqq 2$, *be a partition of* $V \setminus Z$. *If* $G[Z \cup V_i] - e$ *is k-node connected for every edge e in* $G[V_i \cup Z]$ *and for every* $i \in \{1, \cdots, p\}$ *and if* $\hat{G} = (\hat{V}, \hat{E})$ *is 2-node connected then* $\frac{1}{2} \sum_{i=1}^{p} x(\delta_{G-Z}(V_i)) \geqq p - 1$ *defines a facet of* NCON $(G; k)$.

Proof. To prove (b) of Theorem 7.4 we first show that for every spanning tree T of $\hat{G} = (G - Z)/V_1/\cdots/V_p$ the edge set $C \equiv T \cup E'$, where $E' \equiv E(V_1) \cup \cdots \cup E(V_p) \cup E(Z) \cup \delta(Z)$ is k-node connected. By assumption, for every $i \in \{1, \cdots, p\}$ and every two nodes $u, v \in V_i \cup Z$, the graph $G[V_i \cup Z]$ and hence its supergraph (V, C) contains k node disjoint $[u, v]$-paths. Let $u \in V_i$, $v \in V_j$, $i \neq j$. The edge set $C \setminus (E(Z) \cup \delta(Z))$ contains a $[u, v]$-path P by construction. Let t_i be the last node of V_i and t_j the first node of V_j that is encountered by going from u to v along P. Since $G[V_i \cup Z]$ is k-connected it contains a $(u, Z \cup \{t_i\})$-fan and similarly $G[V_j \cup Z]$ contains a $(v, Z \cup \{t_j\})$-fan. This implies that C contains k node disjoint $[u, v]$-paths, and thus C is k-node connected.

Let $\bar{M}$ be the matrix whose columns are the incidence vectors of all spanning trees of $\hat{G}$. Using the arguments of the end of the proof of Theorem 5.6, we can easily show that the 2-node connectedness of $\hat{G}$ implies that $\bar{M}$ has full row rank. Hence $\bar{M}$ contains a nonsingular $s \times s$-submatrix M.

To prove (a) of Theorem 7.4, let $e \in E'$ be an arbitrary edge, say $e \in E(V_1 \cup Z)$. Since e is not k-node-essential with respect to $G[V_1 \cup Z]$, $G[V_1 \cup Z] - e$ is still k-node connected. By assumption and from the arguments used above, it follows that for every spanning tree T of $\hat{G}$, $T \cup E' \setminus \{e\}$ is k-node connected. □

The conditions of Theorem 7.5 are, in particular, satisfied if G is $(k+1)$-node connected and if $G[V_i \cup Z]$ is $(k+1)$-connected for all $i \in \{1, \cdots, p\}$.

Observe that, for $k = 1$, Theorem 7.5 implies that, for any 2-edge connected graph $G = (V, E)$, any partition $V_1, \cdots, V_p$ of V, $p \geqq 2$, such that $G[V_i]$ is 2-edge connected induces a facet defining inequality $\frac{1}{2} \sum_{i=1}^{p} x(\delta(V_i)) \geqq p - 1$ of NCON $(G; 1) =$ CON (G). By Theorem 4.10 these are exactly the facet-defining inequalities of CON (G) of this type. It is easy to see that "$G[V_i]$ is 2-edge connected for $i = 1, \cdots, p$" is a necessary condition for an inequality of type (7.2) (ii) to define a facet of NCON $(G; k)$. However, this condition is far from being sufficient for $k \geqq 2$.

In fact, the connectivity conditions on $G[V_i \cup Z]$ in (7.5) cannot be weakened further. For instance, we can show that for every $k \geqq 1$ there exists a $(k+1)$-node connected graph $G = (V, E)$ with node sets $Z, V_1, \cdots, V_p \subseteq V$ such that $|Z| = k - 1$; $V_1, V_2, \cdots, V_p$ is a partition of V and such that $G[Z]$ is a complete graph, $G[Z \cup V_i]$ and $G[V_i]$ are k-node connected for $i = 1, \cdots, p$ and the corresponding inequality $\frac{1}{2} \sum_{i=1}^{p} x(\delta_{G-Z}(V_i)) \geqq p - 1$ does not define a facet of NCON $(G; k)$. We describe this construction for the case where k is even in the following example.

Example 7.6. Let $k = 2r$, $r \geqq 1$. We construct a graph $G = (V, E)$ with $5k - 1$ nodes as follows. Let A_i, $i = 1, \cdots, 8$ and Z be node sets with the following properties $V = Z \cup \bigcup_{i=1}^{8} A_i$, $|Z| = k - 1$ and Z induces a complete subgraph of G. All A_i are stable sets of G and satisfy $|A_i| = r$. Every node of Z is linked to every node of $A_1 \cup A_2 \cup A_5 \cup A_6$ by an edge.

Every node of A_i is linked to every node of A_j $(i < j)$ by an edge for $j = i + 1$ $(i = 1, 2, 3, 5, 6, 7)$, $i = 1$ and $j = 4$, $i = 5$ and $j = 8$, $i = 3$ and $j = 7$, $i = 4$ and $j = 8$. The scheme of the graph G obtained this way is displayed in Fig. 1.

It is easy to see that for $V_1 = \bigcup_{i=1}^{4} A_i$, $V_2 = \bigcup_{i=5}^{8} A_i$ the graphs $G[V_1]$, $G[V_2]$, $G[Z \cup V_1]$, $G[Z \cup V_2]$ are k-node connected and that G is $(k + r)$-node connected. But, for every edge e with one endnode in A_7 and the other in A_8, or one endnode in A_3 and

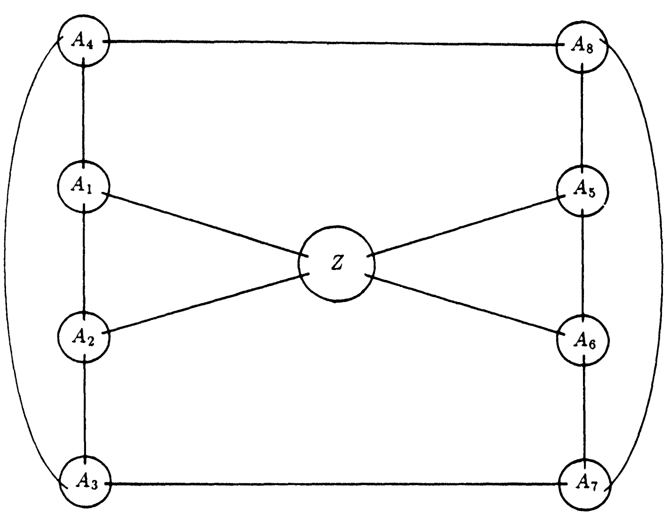

FIG. 1

the other in A_4, there is no spanning tree T in $\hat{G} = (G - Z)/V_1/V_2$ (i.e., T is just an edge) such that $T \cup E(V_1) \cup E(V_2) \cup E(Z) \cup \delta(Z)\setminus\{e\}$ is k-node connected. So by Theorem 7.4 the corresponding inequality does not define a facet of NCON $(G; k)$.

We will now give, for every $k \geqq 2$, an example where no two nodes in $G[Z]$ are adjacent and where $\kappa(G[Z \cup V_i]) = 1$, yet the corresponding inequality defines a facet of NCON $(G; k)$.

Example 7.7. Let $k \geqq 2$. We construct a graph $G = (V, E)$ with $p + k - 1$ nodes, where $p \geqq k + 1$ as follows. Let $Z \subseteq V$ with $|Z| = k - 1$ be a node set such that no two nodes in Z are adjacent. Let $V' \equiv \{v_1, \cdots, v_p\}$ be further $p \geqq k + 1$ nodes and set $V_i \equiv \{v_i\}$, $i = 1, \cdots, p$. The subgraph of G induced by V' is complete and every node of V' is linked to every node of Z by an edge. Thus $G[Z \cup V_i]$ is a star and $\kappa(G[Z \cup V_i]) = 1$. It is easy to see that G is $(k + 1)$-connected and that properties (a) and (b) of Theorem 7.4 are satisfied.

The results of this section show that dropping the integrality stipulations (iv) from (7.2) yields a reasonable (and, for small fixed k, computationally tractable) LP-relaxation for the problem of finding a spanning k-connected subgraph of minimum cost.

8. Node and edge connectivity mixed: A model used in practice. We will now introduce a model that describes the current situation in the area of building fiber optical networks faithfully—see Monma and Shallcross (1989) for a detailed overview of the approaches to design "survivable" communication networks. This model contains a mixture of certain node and edge connectivity requirements. As far as we can see, graphs with these kinds of "mixed connectivity" properties have not received much attention in the graph theory literature nor have the related optimization problems been considered seriously in combinatorial optimization.

As before we begin with the graph $G = (V, E)$ that describes the possible direct connections between the given locations of switches. (Recall that two nodes $s, t \in V$ are called (*locally*) *k-connected* or (*locally*) *k-edge connected* if G contains k $[s, t]$-paths that are node-disjoint or edge-disjoint, respectively.) In the network design area it is common to classify the locations by "type." So, for each node $s \in V$ we introduce two connectivity parameters, denoted by r_s and k_s; r_s is called the *edge connectivity type*, k_s the *node connectivity type*. As usual we have a cost c_e for all $e \in E$. We are looking for a subset C of the edge set E of minimum cost $c(C)$ such that for every two nodes $s, t \in V$, $s \neq t$, (V, C) contains $\min\{r_s, r_t\}$ edge-disjoint $[s, t]$-paths and $\min\{k_s, k_t\}$ node-disjoint $[s, t]$-paths.

The conditions stated above require that the "cable network" (V, C) is locally $\min\{r_s, r_t\}$-edge connected and locally $\min\{k_s, k_t\}$-connected. Since local k-connectedness implies local k-edge connectedness we may assume that

$$(8.1) \qquad r_s \geqq k_s \quad \text{for all } s \in V.$$

Setting, for all $s, t \in V$, $s \neq t$, $r_{st} \equiv \min\{r_s, r_t\}$, $d_{st} \equiv 1$, and $k_{st} \equiv \min\{k_s, k_t\} - 1$, we see that the problem defined above can be viewed as a special case of (2.1).

In telephone network applications of the type considered here, we typically have

$$(8.2) \qquad r_s, k_s \in \{0, 1, 2\} \quad \text{for all } s \in V.$$

(But we also know of a communication network application with $k_s \in \{0, 1, \cdots, 5\}$.) The nodes $s \in V$ with $r_s = k_s = 0$ are the Steiner nodes. They are not required to be in the fiber optical network but they may be used to construct the network. The nodes (respectively, offices or locations) s with $k_s = 2$ are sometimes called "special offices." They frequently carry high loads of communication traffic. Their failure—without the possibility of rerouting—would be fatal to the system and result in considerable losses (financially and in customer good-will).

Using (2.1) and (7.2) we can formulate the mixed connectivity problem described above as an integer linear program as follows:

(8.3) $\min c^T x$

(i) $x(\delta(W)) \geqq \min\{r_s, r_t\}$ for all $s, t \in V$, $s \neq t$, and for all $W \subseteq V$, $s \in W$, $t \notin W$;

(ii) $\frac{1}{2}\sum_{i=1}^{p} x(\delta(V_i)) \geqq p-1$ for all $s, t \in V$, $s \neq t$ and for all $Z \subseteq V \setminus \{s, t\}$, $|Z| = \min\{k_s, k_t\} - 1$, and all nontrivial partitions $V_1, \cdots, V_p$ of $V \setminus Z$, $p \geqq 2$;

(iii) $0 \leqq x_e \leqq 1$ for all $e \in E$;

(iv) $x_e \in \{0, 1\}$ for all $e \in E$.

Needless to say, (8.3) belongs to the class of NP-hard optimization problems; in fact, this is true even in the case where k and r satisfy (8.1) and (8.2), since the Steiner tree problem is a special case.

As before, to address optimization issues it is natural to introduce a polytope associated with the integral solutions of (8.3). So, let $G = (V, E)$ be a graph and $k \in \mathbf{R}^V$, $r \in \mathbf{R}^V$ be two vectors of nonnegative integers. Then

$$\mathrm{CON}_{r,k}(G) \equiv \mathrm{conv}\{x \in \mathbf{R}^E \mid x \text{ satisfies } (8.3)(\mathrm{i}), \cdots, (\mathrm{iv})\}$$

is the convex hull of the incidence vectors χ^C of edge sets $C \subseteq E$ such that for every two nodes $s, t \in V$, $s \neq t$, the subgraph (V, C) is locally $\min\{k_s, k_t\}$-connected and locally $\min\{r_s, r_t\}$-edge connected.

As mentioned before, the separation problems for the inequality systems of (8.3) can be solved in polynomial time, for r_s and k_s small, with not so small degrees of the polynomials of the running time functions, however. To get practically efficient cutting plane algorithms for the solution of (8.3) we have to use heuristic separation routines in addition.

The theoretical work presented here on (partial) characterizations of facets will be used to design fast heuristic separation algorithms. Our work on polyhedral properties of $\mathrm{CON}_{r,k}(G)$, as well as on computational aspects of the LP-relaxation that follows from (8.3), is still in progress and will be reported in a forthcoming paper.

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